On dynamics of geometrically thin accretion disks

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Abstract

Axisymmetric accretion disks in vicinity of a central compact body are studied. For the simple models such as vertically isothermal disks as well as adiabatic ones the exact solutions to the steady-state MHD (magnetohydrodynamic) system were found under the assumption that the radial components of velocity and magnetic field are negligible. On the basis of the exact solution one may conclude that vertically isothermal disks will be totally isothermal. The exact solution for the case of adiabatic disk corroborates the view that thin disk accretion must be highly nonadiabatic. An intermediate approach, that is between the above-listed two, for the modeling of thin accretion disks is developed. In the case of non-magnetic disk, this approach enables to prove, with ease, that all solutions for the midplane circular velocity are unstable provided the disk is non-viscous. Hence, this approach enables to demonstrate that the pure hydrodynamic turbulence in accretion disks is possible. It is interesting that a turbulent magnetic disk tends to be Keplerian. This can easily be shown by assuming that the turbulent gas tends to flow with minimal losses, i.e. to have the Euler number as small as possible.

1 Introduction

We will consider the dynamics of axisymmetric accretion disk around a compact object. A successful theory of the process in question is mainly developed (see, e.g. [5], [7], [22], [20], [21]). Extensive use is made of simple models such as vertically isothermal disks as well as adiabatic ones. To estimate possible errors and limits associated to these models, the exact solutions to the steady-state MHD (magnetohydrodynamic) system were found (see Sec. 2 and Sec. 3) under the assumption that the radial components of velocity and magnetic field are

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negligible. We will also consider an intermediate case that is between the above-listed two axisymmetric flows (see Sec. 4). Such an approach for the modeling of thin accretion disks turns out to be more flexible and efficient. In particular, the question of pure hydrodynamic turbulence is still an open question [7]. The possibility for finite disturbances to develop turbulence in the nonlinear regime was demonstrated by O. A. Kuznetsov in [8] and in doing so he has disproved the well-known arguments that pure hydrodynamic turbulence cannot be a self-sustaining source of viscosity in accretion disks (see, e.g., [7] and references therein). Using the approach of Sec. 4 we will also demonstrate that the pure hydrodynamic turbulence in accretion disks is possible.

The input system of magnetohydrodynamic (MHD) equations is the following (see, e.g. [9, p. 126], [14], [15], [16], [17], [19], [21]):

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot (\rho \mathbf{v}) = 0, \tag{1}$$

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \boldsymbol{\nabla}\cdot \left[\rho \mathbf{v}\mathbf{v} + \left(P + \frac{B^2}{8\pi}\right)\mathbf{I} - \frac{1}{4\pi}\mathbf{B}\mathbf{B}\right] = \nabla \cdot \boldsymbol{\tau} - \rho \nabla \Phi, \tag{2}$$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot \left[\mathbf{v}\left(E + P + \frac{B^2}{8\pi}\right) - \frac{1}{4\pi}\mathbf{B}\left(\mathbf{v}\cdot\mathbf{B}\right)\right] = -\rho \mathbf{v}\cdot\nabla\Phi - \rho\dot{Q} - \nabla\cdot\mathbf{q}, \tag{3}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{4}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{5}$$

$$\mathbf{E} = -\frac{1}{c}(\mathbf{v} \times \mathbf{B}), \tag{6}$$

where ρ, $\mathbf{v}$, P, $\mathbf{B}$, $\mathbf{E}$, $\boldsymbol{\tau}$, $\mathbf{q}$, and Φ denote the density, velocity, pressure, magnetic induction field, electric field, stress tensor, heat current, and gravitational potential, respectively, $B = |\mathbf{B}|$, $v = |\mathbf{v}|$, $E = \rho e_p + 0.5\rho v^2 + B^2/(8\pi)$ denotes the total energy per unit volume with e_p being the internal energy per unit mass for the plasma, $\dot{Q}$ denotes the local cooling rate [5, p. 142]. It is, mainly, assumed that

$$\Phi = -G\frac{M}{\sqrt{r^2 + z^2}}, \qquad G, M = const. \tag{7}$$

The heat conductive flux, $\mathbf{q}$, may be expressed as

$$\mathbf{q} = -\lambda_T \nabla \cdot T, \tag{8}$$

where λ_T denotes the thermal conductivity, T denotes the temperature. The stress tensor, $\boldsymbol{\tau}$, is the sum of two tensors, $\boldsymbol{\tau} = \boldsymbol{\tau}_v + \boldsymbol{\tau}_t$, namely, the viscous, $\boldsymbol{\tau}_v$, and the turbulent, $\boldsymbol{\tau}_t$, stress tensors:

$$\boldsymbol{\tau}_v \approx \mu_v \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^*\right] - \frac{2}{3}\mu_v \nabla \cdot \mathbf{v}\mathbf{I}, \tag{9}$$

$$\boldsymbol{\tau}_t \approx \mu_t \left[\nabla \mathbf{v} + (\nabla \mathbf{v})^*\right] - \frac{2}{3}\left(\mu_t \nabla \cdot \mathbf{v} + \rho\overline{\kappa}\right)\mathbf{I}, \tag{10}$$

where $(\ )^*$ denotes a conjugate tensor, μ_v denotes the dynamic viscosity, μ_t and $\overline{\kappa}$ denote the turbulent viscosity and the kinetic energy of turbulence [1], respectively. We will also use the viscosity $\mu = \mu_v + \mu_t$. Obviously, if the flow is laminar, then $\overline{\kappa} = 0$ and μ is the dynamic viscosity.

We will not take into consideration the energy equation, since the following three axisymmetric flows in cylindrical coordinates, (r, φ, z), will be considered.

1) A vertically isothermal disk, where the temperature is a pre-assigned value,

$$P = \rho RT, \quad \frac{\partial T}{\partial z} = 0, \qquad R = const. \tag{11}$$

2) An adiabatic disk, i.e.

$$P = K\rho^{\gamma}, \qquad \gamma, K = const. \tag{12}$$

3) An intermediate case that is between the above-listed two, in some measure opposite, axisymmetric flows.

Let us note that the number densities of ions and electrons at any point are approximately equal, and, hence, a plasma must always be close to charge neutrality. Even a small charge imbalance would create huge electric fields which would move the plasma particles so as to restore neutrality very quickly [9], [7]. The plasma maintains charge neutrality to a high degree of accuracy. However, local charge imbalances may be produced by thermal fluctuations [9]. To estimate their size, the Debye length is usually used [5], [9], [7]. The length scale of plasma dynamics must be much larger than the Debye length. For example, the mean density of gas in the Milky Way is a million per cubic metre [5]. Then, assuming that the gas temperature is close to $100^\circ\ K$, we find that the Debye length will be less than $0.7\ m$. We will consider the flow at the periphery of accretion disk and, hence, the length scale of plasma dynamics will be much larger than the Debye length. If, however, inside the disk there is a charge density, then it gives rise to an electric field outside the disc which is available to pull charges out of the disc [7]. Hence, in the case of at least steady-state flow, we may write that

$$\nabla \cdot \mathbf{E} = 0 \Rightarrow \nabla \cdot (\mathbf{v} \times \mathbf{B}) = 0. \tag{13}$$

Let us introduce the following characteristic quantities: t_*, l_*, ρ_*, v_*, p_*, T_*, μ_*, $\overline{\kappa}_*$, and B_* for, respectively, time, length, density, velocity, pressure, temperature, viscosity, kinetic energy of turbulence, and magnetic field. The following notation will also be used:

$$S_h = \frac{l_*}{v_* t_*},\ E_u = \frac{p_*}{\rho_* v_*^2},\ \beta = \frac{4\pi p_*}{B_*^2},\ F_r = \frac{v_*^2 l_*}{GM},\ R_e = \frac{\rho_* v_* l_*}{\mu_*},\ \vartheta_{ke} = \frac{2\overline{\kappa}_*}{3v_*^2}, \tag{14}$$

where S_h, E_u, F_r, and R_e denote, respectively, Strouhal, Euler, Froude, and Reynolds numbers. For axisymmetrical flow, we have, in view of (14), the

following non-dimensional system:

$$S_h \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial\left(r\rho v_r\right)}{\partial r} + \frac{\partial\left(\rho v_z\right)}{\partial z} = 0, \tag{15}$$

$$\begin{gathered} S_h \frac{\partial \rho v_r}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r} r\left(\rho v_r^2 - \frac{E_u}{\beta}B_r^2\right) + \frac{\partial}{\partial z}\left(\rho v_r v_z - \frac{E_u}{\beta}B_r B_z\right) + \\ \frac{E_u}{\beta}\frac{B_\varphi^2}{r} - \frac{\rho v_\varphi^2}{r} = -\frac{\partial}{\partial r}\left(E_u P + \frac{E_u}{\beta}\frac{B^2}{2}\right) - \frac{\rho}{F_r}\frac{\partial \Phi}{\partial r} + \\ \frac{1}{R_e}\left\{\frac{\partial}{\partial r}\left[2\mu\frac{\partial v_r}{\partial r} - \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial\left(r v_r\right)}{\partial r} + \frac{\partial v_z}{\partial z}\right)\right] + \right. \\ \left. \frac{\partial}{\partial z}\mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right) + \frac{2\mu}{r}\left(\frac{\partial v_r}{\partial r} - \frac{v_r}{r}\right)\right\} - \vartheta_{ke}\frac{\partial}{\partial r}\rho\overline{\kappa}, \end{gathered} \tag{16}$$

$$\begin{gathered} S_h \frac{\partial \rho v_\varphi}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r} r\left(\rho v_\varphi v_r - \frac{E_u}{\beta}B_\varphi B_r\right) + \\ \frac{\partial}{\partial z}\left(\rho v_\varphi v_z - \frac{E_u}{\beta}B_\varphi B_z\right) + \frac{\rho v_\varphi v_r}{r} - \frac{E_u}{\beta}\frac{B_\varphi B_r}{r} = \\ \frac{1}{R_e}\left\{\frac{\partial}{\partial r}\left[\mu r\frac{\partial}{\partial r}\left(\frac{v_\varphi}{r}\right)\right] + \frac{\partial}{\partial z}\left(\mu\frac{\partial v_\varphi}{\partial z}\right) + 2\mu\frac{\partial}{\partial r}\left(\frac{v_\varphi}{r}\right)\right\}, \end{gathered} \tag{17}$$

$$\begin{gathered} S_h \frac{\partial \rho v_z}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r} r\left(\rho v_z v_r - \frac{E_u}{\beta}B_z B_r\right) + \frac{\partial}{\partial z}\left(\rho v_z^2 - \frac{E_u}{\beta}B_z^2\right) = \\ -\frac{\partial}{\partial z}\left(E_u P + \frac{E_u}{\beta}\frac{B^2}{2}\right) - \frac{\rho}{F_r}\frac{\partial \Phi}{\partial z} + \\ \frac{1}{R_e}\left\{\frac{\partial}{\partial r}\left[\mu\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)\right] + \frac{\partial}{\partial z}\left[2\mu\frac{\partial v_z}{\partial z} - \frac{2}{3}\mu\left(\frac{1}{r}\frac{\partial\left(r v_r\right)}{\partial r} + \frac{\partial v_z}{\partial z}\right)\right] + \right. \\ \left. \frac{\mu}{r}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right)\right\} - \vartheta_{ke}\frac{\partial}{\partial z}\rho\overline{\kappa}, \end{gathered} \tag{18}$$

$$S_h \frac{\partial B_r}{\partial t} + \frac{\partial\left(v_z B_r - v_r B_z\right)}{\partial z} = 0, \tag{19}$$

$$S_h \frac{\partial B_\varphi}{\partial t} + \frac{\partial\left(v_r B_\varphi - v_\varphi B_r\right)}{\partial r} + \frac{\partial\left(v_z B_\varphi - v_\varphi B_z\right)}{\partial z} = 0, \tag{20}$$

$$S_h \frac{\partial B_z}{\partial t} + \frac{1}{r}\frac{\partial r\left(v_r B_z - v_z B_r\right)}{\partial r} = 0, \tag{21}$$

$$\frac{1}{r}\frac{\partial r B_r}{\partial r} + \frac{\partial B_z}{\partial z} = 0. \tag{22}$$

It is, mainly, assumed that

$$\Phi = -\frac{1}{\sqrt{r^2 + z^2}}. \tag{23}$$

We will consider, in general, accretion disks. Hence, it is assumed that the values ρ, v_r, v_φ, P, Φ are even functions of z, whereas v_z is an odd one. In such a case, in view of (15)-(23), there exist two possibilities: 1) B_z is an even function of z, whereas B_φ and B_r are odd ones; 2) the values B_φ and B_r are even functions of z, whereas B_z is an odd one. We will consider the first possibility.

If $v_r = v_z = 0$, then there exists the third possibility, namely, the values B_φ and B_z are even functions of z, whereas B_r is an odd one. In such a case the magnetic field will be unstable provided $B_\varphi \neq 0$. Actually, if $v_r \neq 0$, then B_φ will be an odd function of z, since we consider the case when B_z is an even function of z. It is very important to note that the solution such that B_φ $(\neq 0)$ is an even functions of z can not be obtained as a limiting case (namely, as $v_r \to 0$) of the motion under $v_r \neq 0$. Hence, any solution for B_φ $(\neq 0)$ such that it is not an odd function of z may be seen as unstable, as any infinitesimal variation, δv_r $(\neq 0)$, gives rise to a finite response in the magnetic field.

In the case of steady-state flow, we mainly assume that the flow is charge-neutral [21], i.e.

$$\frac{\partial r\left(v_\varphi B_z - v_z B_\varphi\right)}{r\partial r} + \frac{\partial\left(v_r B_\varphi - v_\varphi B_r\right)}{\partial z} = 0. \tag{24}$$

2 Vertically isothermal disk

In this section we construct steady-state solutions for the system (15)-(23) provided that

$$v_r = 0, \quad \frac{\partial T}{\partial z} = 0. \tag{25}$$

Let us note that the equality $v_r = 0$ implies $v_z = 0$. It could be easily seen from the steady-state version of (15). We take $p_* = \rho_* R T_*$ and, hence, we obtain from (11) that

$$P = \rho T. \tag{26}$$

The temperature in (26) is assumed to be a preassigned function of r.

We intend to find an exact steady-state solution to the MHD system (15)-(22) provided $R_e \to \infty$, $\vartheta_{ke} = 0$, and

$$B_r = 0. \tag{27}$$

In such a case, the MHD system is reduced to the following.

$$\frac{E_u}{\beta}\frac{B_\varphi^2}{r} - \frac{\rho v_\varphi^2}{r} = -\frac{\partial}{\partial r}\left(E_u P + \frac{E_u}{\beta}\frac{B^2}{2}\right) - \frac{\rho}{F_r}\frac{\partial \Phi}{\partial r}, \tag{28}$$

$$\frac{\partial\left(B_\varphi B_z\right)}{\partial z} = 0, \tag{29}$$

$$-\frac{\partial}{\partial z}\left(\frac{E_u}{\beta}B_z^2\right) = -\frac{\partial}{\partial z}\left(E_u P + \frac{E_u}{\beta}\frac{B^2}{2}\right) - \frac{\rho}{F_r}\frac{\partial \Phi}{\partial z}, \tag{30}$$

$$\frac{\partial\left(v_{\varphi}B_{z}\right)}{\partial z}=0, \tag{31}$$

$$\frac{\partial B_{z}}{\partial z}=0. \tag{32}$$

In view of (32), (31), and (29), we obtain:

$$B_{z}=B_{z}\left(r\right),\ v_{\varphi}=v_{\varphi}\left(r\right),\ B_{\varphi}=B_{\varphi}\left(r\right). \tag{33}$$

Then we obtain, instead of (28)-(32):

$$\frac{E_{u}}{\beta}\frac{B_{\varphi}^{2}}{r}-\frac{\rho v_{\varphi}^{2}}{r}=-\frac{\partial}{\partial r}\left(E_{u}P+\frac{E_{u}}{\beta}\frac{B_{z}^{2}+B_{\varphi}^{2}}{2}\right)-\frac{\rho}{F_{r}}\frac{\partial\Phi}{\partial r}, \tag{34}$$

$$0=-\frac{\partial}{\partial z}\left(E_{u}P\right)-\frac{\rho}{F_{r}}\frac{\partial\Phi}{\partial z}. \tag{35}$$

Let

$$C_{\rho}=\left.\rho\right|_{z=0},\quad \phi=\left.\Phi\right|_{z=0}, \tag{36}$$

By virtue of (26), we find from (35) that

$$\rho=C_{\rho}\exp\left(\frac{\phi-\Phi}{TF_{r}E_{u}}\right),\quad C_{\rho}=C_{\rho}\left(r\right),\ \phi=\phi\left(r\right)\equiv\left.\Phi\right|_{z=0}. \tag{37}$$

If (23) is valid, then

$$\rho=C_{\rho}\exp\left[\frac{1}{TF_{r}E_{u}}\left(\frac{1}{\sqrt{r^{2}+z^{2}}}-\frac{1}{r}\right)\right],\quad C_{\rho}=C_{\rho}\left(r\right). \tag{38}$$

Let, in general,

$$\frac{\partial\Phi}{\partial z}\neq 0. \tag{39}$$

Eq. (34) must be valid under all values of $z\geq 0$. After differentiation (34) over z, in view of (35) we obtain

$$\frac{v_{\varphi}^{2}}{r}\frac{\rho}{TE_{u}}=\frac{\partial\rho}{\partial r}+\frac{\rho}{TF_{r}E_{u}}\frac{\partial\Phi}{\partial r}. \tag{40}$$

Since, in view of (37),

$$\frac{\partial\rho}{\partial r}=\left[\frac{\partial C_{\rho}}{\partial r}+C_{\rho}\frac{\partial}{\partial r}\left(\frac{\phi-\Phi}{TF_{r}E_{u}}\right)\right]\exp\left(\frac{\phi-\Phi}{TF_{r}E_{u}}\right), \tag{41}$$

we find from (40) that

$$\frac{v_{\varphi}^{2}}{r}\frac{C_{\rho}}{TE_{u}}=\frac{\partial C_{\rho}}{\partial r}+\frac{C_{\rho}}{TF_{r}E_{u}}\frac{\partial\phi}{\partial r}-\frac{C_{\rho}\left(\phi-\Phi\right)}{T^{2}F_{r}E_{u}}\frac{\partial T}{\partial r}. \tag{42}$$

After differentiation (42) with respect to z, we obtain:

$$\frac{C_\rho}{T^2 F_r E_u}\frac{\partial \Phi}{\partial z}\frac{\partial T}{\partial r} = 0. \tag{43}$$

Since $C_\rho \neq 0$, we obtain from (43), in view of (39), that

$$T = const, \tag{44}$$

and, hence,

$$C_\rho = const \exp\left(\int \frac{v_\varphi^2}{rTE_u} dr - \frac{\phi}{TF_rE_u}\right). \tag{45}$$

If (23) is valid, then $\phi = -1\diagup r$. Let the disk be Keplerian, i.e.

$$v_\varphi = \frac{1}{\sqrt{rF_r}}, \tag{46}$$

then

$$C_\rho = const \exp\left(\int \frac{dr}{r^2TF_rE_u} + \frac{1}{rTF_rE_u}\right) = const. \tag{47}$$

Thus, the assumption that the motion is Keplerian leads to a constant density at the midplane. Let us consider a vortex motion, i.e.

$$v_\varphi = \frac{C_\varphi}{r}, \quad C_\varphi = const, \tag{48}$$

and let, for the sake of simplicity, $C_\varphi = 1\diagup\sqrt{F_r}$, then, in general, we have

$$\frac{\partial C_\rho}{\partial r} < 0. \tag{49}$$

The density at the midplane, in view of (45), will be the following.

$$C_\rho = const \exp\left(\frac{1}{rTF_rE_u} - \frac{1}{2r^2TF_rE_u}\right). \tag{50}$$

Substituting (37), (44), and (45) into (34), we obtain the following equation in B_φ and B_z.

$$\frac{B_\varphi^2}{r} + \frac{\partial}{\partial r}\left(\frac{B_z^2 + B_\varphi^2}{2}\right) = 0. \tag{51}$$

Let us assume that the flow is electrically neutral [21] to close Eq. (51). Hence

$$\frac{\partial r\left(v_\varphi B_z\right)}{r\partial r} = 0. \tag{52}$$

If we have a vortex flow, i.e. (48) is valid, then

$$B_z = const. \tag{53}$$

By virtue of (53), we find from (51):

$$B_\varphi = \frac{C_{b\varphi}}{r}, \quad C_{b\varphi} = const. \tag{54}$$

Let us note that the solution, (54), for B_φ is unstable, as this solution is an even function of z (see Sec. 1), and, hence, $B_\varphi = 0$.

Conversely, we can see from (33) that B_φ ($\neq 0$) is an even function of z, and, hence, B_φ is unstable (see Sec. 1). Thus, the stable solution is: $B_\varphi = 0$, and, by virtue of (51), we find that (53) is valid. Assuming that the flow is electrically neutral [21], we find from (52) that (48) is valid, i.e. we have the vortex flow.

The semi-thickness, H, of disk is often (e.g. [6]) defined as

$$H = \frac{1}{C_\rho} \int\limits_0^\infty \rho dz, \quad C_\rho = \left. \rho \right|_{z=0}. \tag{55}$$

Notice, using the exact solution (38) in (55) we find that $H \to \infty$ provided that $TF_rE_u \neq 0$. Thus, even if the value of E_u be small but finite, the disk cannot be thin in terms of (55). Instead of the exact solution, (38), it can be used the following approximation (Cf. [18], [21]) for small values of z.

$$\rho \approx C_\rho \exp\left(-\frac{z^2}{2TF_rE_ur^3}\right) \equiv C_\rho \exp\left[-\frac{1}{2}\left(z/H\right)^2\right], \quad H = r\sqrt{rTF_rE_u}. \tag{56}$$

Let us note that the semi-thickness $H \propto r\sqrt{Tr}$ in (56). Analogous formulae can be found in many monographes (see, e.g., [6], [18], [21] and references therein). Thus, in the case of isothermal flow we have

$$H \propto r^{1.5} \tag{57}$$

3 Adiabatic flow

In this section we intend to find an exact steady-state solution to the MHD system (15)-(22) provided $R_e \to \infty$, $\vartheta_{ke} = 0$, and

$$v_r = 0, \quad B_r = 0. \tag{58}$$

Let us remind that the equality $v_r = 0$ implies $v_z = 0$. It could be easily seen from the steady-state version of (15). We take $p_* = K\rho_*^\gamma$ and, hence, we obtain from (12) that

$$P = \rho^\gamma. \tag{59}$$

Then, by analogy with Sec. 2, we obtain:

$$B_z = B_z(r), \ v_\varphi = v_\varphi(r), \ B_\varphi = B_\varphi(r). \tag{60}$$

$$\frac{E_u}{\beta}\frac{B_\varphi^2}{r} - \frac{\rho v_\varphi^2}{r} = -\frac{\partial}{\partial r}\left(E_uP + \frac{E_u}{\beta}\frac{B_z^2 + B_\varphi^2}{2}\right) - \frac{\rho}{F_r}\frac{\partial \Phi}{\partial r}, \tag{61}$$

$$0 = -\frac{\partial}{\partial z}\left(E_u P\right) - \frac{\rho}{F_r}\frac{\partial \Phi}{\partial z}. \tag{62}$$

It is assumed that $P\left(r,z\right)_{z=H} = 0$ and, hence, $\rho\left(r,z\right)_{z=H} = 0$, where $2H\left(r\right)$ denotes the height of disk. Let

$$\Psi = \frac{\gamma}{\gamma - 1}\rho^{\gamma-1} \;\Rightarrow\; \frac{1}{\rho}\nabla P = \nabla\Psi. \tag{63}$$

By virtue of (63), we rewrite (62) to read

$$\frac{\partial \Psi}{\partial z} + \frac{1}{F_r E_u}\frac{\partial \Phi}{\partial z} = 0. \tag{64}$$

We find from (64)

$$\Psi + \frac{\Phi}{F_r E_u} = C\left(r\right). \tag{65}$$

Since $\rho\left(r,z\right)_{z=H} = 0$ and, hence, $\Psi\left(r,z\right)_{z=H} = 0$, we obtain from (65) that

$$\Psi = \frac{1}{F_r E_u}\left(\frac{1}{\sqrt{r^2+z^2}} - \frac{1}{\sqrt{r^2+H^2}}\right). \tag{66}$$

Hence

$$\rho = \left[\frac{\gamma-1}{\gamma F_r E_u}\left(\frac{1}{\sqrt{r^2+z^2}} - \frac{1}{\sqrt{r^2+H^2}}\right)\right]^{1/(\gamma-1)}. \tag{67}$$

By virtue of (61) and (63), we find

$$\frac{E_u}{\beta}\frac{B_\varphi^2}{r} - \frac{\rho v_\varphi^2}{r} = -\rho E_u \frac{\partial \Psi}{\partial r} - \frac{\partial}{\partial r}\left(\frac{E_u}{\beta}\frac{B_z^2 + B_\varphi^2}{2}\right) - \frac{\rho}{F_r}\frac{\partial \Phi}{\partial r}. \tag{68}$$

Eq. (68) must be valid under all values of $z \geq 0$. Since $\rho\left(r,z\right)_{z=H} = 0$, we obtain from (68) that

$$\frac{B_\varphi^2}{r} = -\frac{\partial}{\partial r}\left(\frac{B_z^2 + B_\varphi^2}{2}\right). \tag{69}$$

By virtue of (63) and (69) we obtain from (68):

$$\frac{v_\varphi^2}{r E_u} = \frac{\partial \Psi}{\partial r} + \frac{1}{F_r E_u}\frac{\partial \Phi}{\partial r} \equiv \frac{\partial}{\partial r}\left(\Psi + \frac{\Phi}{F_r E_u}\right) \equiv -\frac{1}{F_r E_u}\frac{\partial}{\partial r}\frac{1}{\sqrt{r^2+H^2}}. \tag{70}$$

In view of (70), we have

$$\frac{v_\varphi^2}{r} = -\frac{1}{F_r}\frac{\partial}{\partial r}\frac{1}{\sqrt{r^2+H^2}}. \tag{71}$$

Let, for instance, the motion is Keplerian, i.e.

$$v_\varphi = \frac{1}{\sqrt{r F_r}}, \tag{72}$$

then, by virtue of (71), we obtain that

$$H^2 = 0. \tag{73}$$

Hence, in view of (67), $\rho = 0$.

We can see from (60) that B_φ ($\neq 0$) is an even function of z, and, hence, B_φ is unstable (see Sec. 1). Thus, the stable solution is: $B_\varphi = 0$, and, by virtue of (69), we find that $B_z = const$. Assuming that the flow is electrically neutral [21], we find from (24) that the flow is the vortex, i.e.

$$v_\varphi = \frac{C_\varphi}{r}. \tag{74}$$

Let, for the sake of simplicity,

$$C_\varphi^2 = \frac{1}{F_r}, \tag{75}$$

then, by virtue of (71), we obtain that

$$H = r\sqrt{4r^2 - 1} \Rightarrow H \approx 2r^2. \tag{76}$$

Thus, in the case of adiabatic flow, we find that

$$H \propto r^2 \tag{77}$$

4 Perfect gas. Pre-assigned midplane temperature

As it can be seen from Sec. 2, the vertically isothermal disk will, in fact, be totally isothermal under the assumption that the radial components, v_r and B_r, of velocity and magnetic field, respectively, are negligible. Furthermore, the disk cannot be thin in terms of, e.g., [6]. Adiabatic disks (see Sec. 3), in contrast to vertically isothermal ones, are more trustworthy. However, thin disk accretion must be highly nonadiabatic, as emphasized in [20]. Because of this, we will consider an intermediate case that is between the above-listed two axisymmetric flows.

In this section we will, mainly, deal with thin disks, i.e. it is assumed that $E_u \ll 1$. We take $p_* = \rho_* RT_*$ and, hence, we obtain from (11) that

$$P = \rho T. \tag{78}$$

It is significant that the temperature at the midplane, i.e.

$$T_0 \equiv T|_{z=0} = T_0\left(r\right), \tag{79}$$

is assumed to be a preassigned function of r.

4.1 Non-magnetic disk

It is assumed that

$$B_r = B_z = B_\varphi = 0. \tag{80}$$

Let $v_z = v_r = 0$, and let $R_e \to \infty$, $\vartheta_{ke} = 0$. In such a case the steady-state version of the system (15)-(23) takes the following form.

$$\frac{\rho v_\varphi^2}{r} = E_u \frac{\partial P}{\partial r} + \frac{\rho}{F_r}\frac{\partial \Phi}{\partial r}, \quad P = \rho T, \quad E_u, F_r = const, \tag{81}$$

$$E_u \frac{\partial P}{\partial z} + \frac{\rho}{F_r}\frac{\partial \Phi}{\partial z} = 0, \quad \Phi \equiv -\frac{1}{\sqrt{r^2+z^2}} = -\frac{1}{r} + \frac{1}{2r^3} z^2 + O\left(z^4\right), \tag{82}$$

Since the values ρ, v_φ, and P are even functions of z, we can write for small values of z:

$$\rho = \rho_0 + \rho_2 z^2 + \ldots, \ T = T_0 + T_2 z^2 + \ldots, \ v_\varphi = v_{\varphi 0} + v_{\varphi 2} z^2 + \ldots, \tag{83}$$

where the coefficients ρ_0, ρ_2, T_0, T_2, $v_{\varphi 0}$, and $v_{\varphi 2}$ depend on r, only. Let us note that namely $z/r \ll 1$ and, hence, the power series in (83) must be represented in z/r. However, for simplicity sake, the denominators (r^i, $i = 0, 2, \ldots$) are, manely, included into the coefficients.

Let

$$\rho|_{z=H} = 0, \ \ T|_{z=H} = 0. \tag{84}$$

Then, by virtue of (82)-(84), we obtain the following equalities:

$$T_0 \rho_2 + T_2 \rho_0 + \frac{\rho_0}{2E_u F_r r^3} = 0, \tag{85}$$

$$T_0 + T_2 H^2 \approx 0, \tag{86}$$

$$\rho_0 + \rho_2 H^2 \approx 0. \tag{87}$$

By virtue of (85)-(87), we find:

$$H^2 \approx 4 T_0 E_u F_r r^3, \tag{88}$$

$$T = T_0 - \frac{1}{4E_u F_r r^3} z^2 + O\left(z^4\right) \equiv T_0 - \frac{1}{4E_u F_r r}\left(\frac{z}{r}\right)^2 + O\left((z/r)^4\right), \tag{89}$$

$$\rho = \rho_0 \left(1 - \frac{z^2}{H^2}\right) + O\left(z^4\right) \equiv \rho_0 - \frac{\rho_0}{4T_0 E_u F_r r}\left(\frac{z}{r}\right)^2 + O\left((z/r)^4\right). \tag{90}$$

By virtue of (83) and (81), we find the following differential equation in the function $\rho_0 = \rho_0(r)$.

$$\frac{\rho_0 v_{\varphi 0}^2}{r} = E_u \frac{\partial T_0 \rho_0}{\partial r} + \frac{\rho_0}{r^2 F_r}, \ r > r_0, \tag{91}$$

where $T_0 = T_0(r)$ is a preassigned function. We will use the following notation $T_0^0 = T_0(r_0)$. The boundary condition is the following

$$\rho_0^0 = \rho_0(r_0). \tag{92}$$

The degenerate equation corresponding to (91), i.e., the equation obtained from (91) by putting $E_u = 0$, will be fulfilled if, and only if, the motion at the midplane is Keplerian:

$$v_{\varphi 0} = \pm \frac{1}{\sqrt{rF_r}}. \tag{93}$$

Notice, any arbitrary function $\rho_0 = \rho_0(r)$ fulfills the degenerate equation and, hence, there is no problem to take a suitable $\rho_0(r)$ such that (92) would be valid and $\rho_0(r) \to 0$ as $r \to \infty$.

Let $E_u > 0$, then the solution to (91), (92) is the following.

$$\rho_0 = \frac{\rho_0^0 T_0^0}{T_0} \exp \int_{r_0}^{r} \frac{1}{rE_u T_0} \left(v_{\varphi 0}^2 - \frac{1}{rF_r} \right) dr. \tag{94}$$

It must be emphasized that ρ_0 is not an analytic function of the small parameter E_u, as it can be easily seen from (94). Such a situation is common for singular systems. Notice, be the disk Keplerian, we would obtain, by virtue of (94), that

$$\rho_0 = \frac{\rho_0^0 T_0^0}{T_0(r)}, \quad \rho_0^0,\, T_0^0 = const. \tag{95}$$

Thus, if $\rho_0(r) \to 0$ as $r \to \infty$, then, in view of (95), $T_0(r) \to \infty$ as $r \to \infty$, and vice versa, i.e. if $T_0(r) \to 0$, then $\rho_0(r) \to \infty$ as $r \to \infty$. Assuming that $T_0 = T_0^0 = const$, we obtain that $\rho_0 = \rho_0^0 = const$ too. Hence, the assumption that the motion is Keplerian leads to improbable density and temperature distributions at the midplane under $r \geq r_0$. To avoid such unlikely solutions, we have to accept that the motion is not Keplerian and, hence

$$v_{\varphi 0}^2 < \frac{1}{rF_r}, \quad r \to \infty. \tag{96}$$

Hence, in view of (94), $\rho_0 \to 0$ as $E_u \to 0$ in contrast to the degenerate solution, where $\rho_0(r)$ is an arbitrary function. Hence, the solution is not a continuous function of the small parameter E_u. It is also a characteristic feature of singular systems.

It is well known (see, e.g., [5], [20], [22]) that a gaseous disk formed around a central star is in an almost Keplerian rotation with a small inward drift velocity. Let us investigate this almost Keplerian rotation. Let $v_r \neq 0$, then, instead of (81)-(82), we should consider the following system.

$$\frac{1}{r}\frac{\partial(r\rho v_r)}{\partial r} + \frac{\partial(\rho v_z)}{\partial z} = 0, \tag{97}$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\left(\rho v_r^2\right)+\frac{\partial}{\partial z}\left(\rho v_r v_z\right)-\frac{\rho v_\varphi^2}{r}=-\frac{\partial}{\partial r}\left(E_u P\right)-\frac{\rho}{F_r}\frac{\partial\Phi}{\partial r}, \tag{98}$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\left(\rho v_\varphi v_r\right)+\frac{\partial}{\partial z}\left(\rho v_\varphi v_z\right)+\frac{\rho v_\varphi v_r}{r}=0, \tag{99}$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\left(\rho v_z v_r\right)+\frac{\partial}{\partial z}\left(\rho v_z^2\right)=-\frac{\partial}{\partial z}\left(E_u P\right)-\frac{\rho}{F_r}\frac{\partial\Phi}{\partial z}. \tag{100}$$

For the sake of simplicity, we take the characteristic quantity, l_*, for length such that

$$r_0=1. \tag{101}$$

Since v_r is an even function of z, but v_z is an odd one, we can write for small values of z:

$$v_r=v_{r0}+v_{r2}z^2+\ldots,\ v_z=v_{z1}z+v_{z3}z^3+\ldots, \tag{102}$$

where the coefficients depend on r, only. By virtue of (83), (102), we obtain from (97), (99) that

$$\frac{1}{r}\frac{\partial\left(r\rho_0 v_{r0}\right)}{\partial r}+\rho_0 v_{z1}=0, \tag{103}$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\left(\rho_0 v_{\varphi 0}v_{r0}\right)+\rho_0 v_{\varphi 0}v_{z1}+\frac{\rho_0 v_{\varphi 0}v_{r0}}{r}=0. \tag{104}$$

By virtue of (103)-(104), we obtain:

$$\rho_0 v_{r0}\frac{\partial r v_{\varphi 0}}{r\partial r}=0. \tag{105}$$

Thus, we find the following exact solution for the midplane value, $v_{\varphi 0}$, of circular velocity:

$$v_{\varphi 0}=\frac{C_{\varphi 0}}{r},\quad C_{\varphi 0}=const. \tag{106}$$

For the sake of convenience, let us represent $C_{\varphi 0}$ as a function of the Froude number, F_r. Let the point $r=r_m$ be the only point where the motion is Keplerian, i.e.

$$\frac{C_{\varphi 0}}{r_m}=\frac{1}{\sqrt{r_m F_r}}\Rightarrow C_{\varphi 0}=\sqrt{\frac{r_m}{F_r}}. \tag{107}$$

Hence,

$$v_{\varphi 0}\equiv\left.v_\varphi\right|_{z=0}=\frac{\sqrt{r_m}}{r\sqrt{F_r}},\quad r_m=const. \tag{108}$$

Notice, in the case of a non-viscous flow with $v_{r0}\neq 0$ we have the only solution, (106), for the midplane circular velocity, $v_{\varphi 0}$. If, however, $v_{r0}=0$, then we have infinitely many solutions for $v_{\varphi 0}$. The only limitations are the boundary conditions. In particular, the function $v_{\varphi 0}=v_{\varphi 0}\left(r\right)$ must be such that $\rho_0\left(r\right)\to 0$ as $r\to\infty$ in (94). Thus, if $v_{r0}=0$, then we have the ill-posed problem, i.e. the problem is not well-posed in the sense of Hadamard [1]. It is important

to note that any solution (excluding the vortex) for $v_{\varphi 0}$ can not be obtained as a limiting case (namely, as $v_{r0} \to 0$) of the motion under $v_{r0} \neq 0$. Hence, any solution that does not coincide with the vortex will be unstable, as any infinitesimal variation, $\delta v_{r0} \neq 0$, gives rise to a finite response in the gas flow. Nevertheless, the power-law model [22, p. 374]

$$v_{\varphi 0} = \frac{C_\varphi}{r^{\varkappa}}, \quad \varkappa, C_\varphi = const, \ 0.5 \leq \varkappa < 1, \tag{109}$$

is assumed to be stable (Rayleigh stable [7]) to pure hydrodynamic perturbations, since it satisfies the so-called Rayleigh stability (necessary and sufficient) condition [2, p. 78]:

$$f_R^2 \equiv \frac{1}{r^3} \frac{\partial}{\partial r} \left(r^2 \Omega\right)^2 > 0, \tag{110}$$

where Ω denotes the angular velocity, f_R^2 denotes the Rayleigh frequency.

Let us estimate $v_{\varphi 2}$ in (83). We will use the following asymptotic expansion

$$T_0\left(r\right) \approx \sum_{n=0}^{\infty} a_n r^{-n} \tag{111}$$

in the limit $r \to \infty$. As usually, the series in (111) may converge or diverge, but its partial sums are good approximations to $T_0\left(r\right)$ for large enough r. Assuming that $T_0\left(r\right) \to 0$ as $r \to \infty$, we find that $a_0 = 0$. Then, assuming the characteristic quantity, T_*, for temperature such that $a_1 = 1$, we represent $T_0\left(r\right)$ in the following form:

$$T_0\left(r\right) = \frac{1}{r} + O\left(r^{-2}\right), \quad r \to \infty. \tag{112}$$

Thus, we can use the following approximation:

$$T_0\left(r\right) \approx \frac{1}{r}, \quad r \to \infty. \tag{113}$$

One can easily see, for instance, that the midplane temperature of the adiabatic disks considered in Sec. 3 can be approximated by (113).

By virtue of (113), we find from (88) that

$$H \propto r. \tag{114}$$

Let v_z is a linear function of z, i.e. $v_z = v_{z1} z$ with $v_{z1} = v_{z1}\left(r\right)$. Since

$$\left. v_z \right|_{z=H} = \frac{\partial H}{\partial r} \left. v_r \right|_{z=H}, \tag{115}$$

we find, by virtue of (113) and (88), that

$$v_{z1} H \approx \frac{\partial H}{\partial r} v_{r0} \Rightarrow v_{z1} \approx \frac{v_{r0}}{r}. \tag{116}$$

By virtue of (102) and (83), we obtain from (97), (99) that

$$v_{r0}\frac{\partial v_{\varphi 2}}{\partial r}+2v_{z1}v_{\varphi 2}+\frac{v_{\varphi 2}v_{r0}}{r}=0. \tag{117}$$

Thus, by virtue of (116), we find

$$v_{\varphi 2}\approx\frac{const}{r^3}. \tag{118}$$

By virtue of (83), (88), (113), and (118), we find that

$$v_\varphi|_{z=H}\approx\frac{C_{\varphi 0}}{r}\left(1-C_{\varphi 2}E_uF_r\right),\quad C_{\varphi 2}=const>0. \tag{119}$$

Since $E_u\ll 1$, it can be assumed that

$$v_\varphi\approx\frac{C_{\varphi 0}}{r},\quad 0\le z\le H. \tag{120}$$

It is easy to see from (120) that the Rayleigh frequency $f_R^2=0$ and, hence, the vortex is Rayleigh unstable. Let us consider the problem of vortex stability from another point of view.

4.1.1 Stability

Let $\overline{\varsigma}$ denote the value of a dependent variable ς for the case of a steady-state solution. It is assumed that we consider the motion close to the solution to the system (81)-(82). In view of (108) and (113), the solution can be written in the form:

$$\overline{v}_z=\overline{v}_r=0,\ \overline{v}_{\varphi 0}\equiv\overline{v}_\varphi|_{z=0}=\frac{\sqrt{r_m}}{r\sqrt{F_r}},\quad r_m=const, \tag{121}$$

$$\frac{\overline{\rho}_0}{\rho_0^0}=r\exp\int\limits_1^r\frac{1}{E_u}\left(\overline{v}_{\varphi 0}^2-\frac{1}{rF_r}\right)dr. \tag{122}$$

In such a case the non-linear system in perturbations $\widetilde{\varsigma}$ ($\varsigma=\overline{\varsigma}+\widetilde{\varsigma}$) for (15)-(18) will be, under $R_e\to\infty$, as follows.

$$S_h\frac{\partial\left(\overline{\rho}_0+\widetilde{\rho}_0\right)}{\partial t}+\frac{1}{r}\frac{\partial r\left(\overline{\rho}_0+\widetilde{\rho}_0\right)\widetilde{v}_{r0}}{\partial r}+\left(\overline{\rho}_0+\widetilde{\rho}_0\right)\widetilde{v}_{z1}=0, \tag{123}$$

$$S_h\frac{\partial\left(\overline{\rho}_0+\widetilde{\rho}_0\right)\widetilde{v}_{r0}}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}r\left(\overline{\rho}_0+\widetilde{\rho}_0\right)\widetilde{v}_r^2+\left(\overline{\rho}_0+\widetilde{\rho}_0\right)\widetilde{v}_{r0}\widetilde{v}_{z1}-\frac{\left(\overline{\rho}_0+\widetilde{\rho}_0\right)\left(\overline{v}_{\varphi 0}+\widetilde{v}_{\varphi 0}\right)^2}{r}=$$
$$-E_u\frac{\partial}{\partial r}\left(\overline{\rho}_0+\widetilde{\rho}_0\right)\left(\overline{T}_0+\widetilde{T}_0\right)-\frac{\overline{\rho}_0+\widetilde{\rho}_0}{r^2F_r}, \tag{124}$$

$$S_h\frac{\partial\left(\overline{\rho}_0+\widetilde{\rho}_0\right)\left(\overline{v}_{\varphi 0}+\widetilde{v}_{\varphi 0}\right)}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}r\left(\overline{\rho}_0+\widetilde{\rho}_0\right)\left(\overline{v}_{\varphi 0}+\widetilde{v}_{\varphi 0}\right)\widetilde{v}_{r0}+$$

$$(\overline{\rho}_0 + \widetilde{\rho}_0)(\overline{v}_{\varphi 0} + \widetilde{v}_{\varphi 0})\widetilde{v}_{z1} + \frac{(\overline{\rho}_0 + \widetilde{\rho}_0)(\overline{v}_{\varphi 0} + \widetilde{v}_{\varphi 0})\widetilde{v}_{r0}}{r} = 0, \tag{125}$$

$$S_h \frac{\partial(\overline{\rho}_0 + \widetilde{\rho}_0)\widetilde{v}_{z1}}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r} r(\overline{\rho}_0 + \widetilde{\rho}_0)\widetilde{v}_{z1}\widetilde{v}_{r0} + 2(\overline{\rho}_0 + \widetilde{\rho}_0)\widetilde{v}_{z1}^2 =$$
$$-2E_u \frac{\partial}{\partial z}\left[(\overline{\rho}_0 + \widetilde{\rho}_0)\left(\overline{T}_2 + \widetilde{T}_2\right) + (\rho_2 + \widetilde{\rho}_2)\left(\overline{T}_0 + \widetilde{T}_0\right)\right] - \frac{\overline{\rho}_0 + \widetilde{\rho}_0}{F_r r^3}. \tag{126}$$

Our main purpose is to reduce (123)-(126) to an ODE system for a subsequent stability investigation. We will consider a function as admissible if it will be a singular function [13]. That is, this function is continous on $1 \leq r < \infty$ and the derivative over the coordinate r exists and is zero almost everywhere. It can be also assumed that the function is strictly monotone decreasing, e.g. $\widetilde{v}_r|_{r\to\infty} \to 0$. Cantor staircase [13] and Lebesgue singular function [11], [12] are well known examples of such functions. Singular functions occur in physics, dynamical systems, etc. (see e.g. the references in [11]).

We restrict ourself to the case of linear system. Assuming the non-linear terms in (123)-(126) as negligible, we get:

$$S_h \frac{\partial \widetilde{\rho}_0}{\partial t} + \frac{1}{r}\frac{\partial r\overline{\rho}_0\widetilde{v}_{r0}}{\partial r} + \overline{\rho}_0\widetilde{v}_{z1} = 0, \tag{127}$$

$$S_h \overline{\rho}_0 \frac{\partial \widetilde{v}_{r0}}{\partial t} - \frac{2\overline{\rho}_0\overline{v}_{\varphi 0}\widetilde{v}_{\varphi 0} + \widetilde{\rho}_0\overline{v}_{\varphi 0}^2}{r} =$$
$$-E_u \frac{\partial}{\partial r}\left(\widetilde{\rho}_0\overline{T}_0 + \overline{\rho}_0\widetilde{T}_0\right) - \frac{\widetilde{\rho}_0}{r^2 F_r}, \tag{128}$$

$$\frac{\partial \widetilde{v}_{\varphi 0}}{\partial t} = 0, \tag{129}$$

$$S_h \overline{\rho}_0 \frac{\partial \widetilde{v}_{z1}}{\partial t} = -2E_u\left(\overline{\rho}_0\widetilde{T}_2 + \widetilde{\rho}_0\overline{T}_2 + \rho_2\widetilde{T}_0 + \widetilde{\rho}_2\overline{T}_0\right) - \frac{\widetilde{\rho}_0}{r^3 F_r}. \tag{130}$$

By virtue of (88)-(90), we obtain from (127)-(130):

$$\frac{\partial \widetilde{\rho}_0}{\partial t} = a_{12}\widetilde{v}_{r0} + a_{13}\widetilde{v}_{z1}, \tag{131}$$

$$\frac{\partial v_r}{\partial t} = a_{21}\widetilde{\rho}_0 + b_2, \tag{132}$$

$$\frac{\partial \widetilde{v}_{z1}}{\partial t} = b_3, \tag{133}$$

where

$$a_{12} = -\frac{\partial r\overline{\rho}_0}{rS_h\partial r},\ a_{13} = -\frac{\overline{\rho}_0}{S_h},\ a_{21} = \frac{C_{\varphi 0}^2}{r^3 S_h\overline{\rho}_0} - E_u\frac{\partial\overline{T}_0}{S_h\overline{\rho}_0\partial r} - \frac{1}{\overline{\rho}_0 r^2 S_h F_r}, \tag{134}$$

$$b_2 = \frac{2C_{\varphi 0}}{r^2 S_h}\widetilde{v}_{\varphi 0} - E_u\frac{\partial\overline{\rho}_0}{S_h\overline{\rho}_0\partial r}\widetilde{T}_0,\ b_3 = -E_u\frac{2\widetilde{T}_2}{S_h}. \tag{135}$$

Hence

$$\frac{\partial^2 \widetilde{\rho}_0}{\partial t^2} - a_{12} a_{21} \widetilde{\rho}_0 = a_{12} b_2 + a_{13} b_3. \tag{136}$$

Obviously, Eq. (136) and, hence, the system (131)-(133) will be unstable if $a_{12} a_{21} > 0$.

By virtue of (122) and (113), we find

$$\frac{\overline{\rho}_0}{\rho_0^0} = r^{1-\eta} \exp\left(\frac{C_{\varphi 0}^2}{E_u} r^{-1} - \frac{C_{\varphi 0}^2}{E_u} \right), \quad \eta = \frac{1}{E_u F_r} > 1,\ E_u F_r < 1. \tag{137}$$

In such a case we obtain that $a_{12} a_{21} > 0$ if

$$r > \max\left(\frac{F_r C_{\varphi 0}^2}{1 - E_u F_r}, \frac{C_{\varphi 0}^2 F_r}{(2 E_u F_r - 1)} \right). \tag{138}$$

Since the disk is thin, i.e. $E_u \ll 1$, and using (107), we find the following sufficient condition of linear instability

$$r > \frac{r_m}{1 - E_u F_r}, \tag{139}$$

where r_m is the only point at which the motion is Keplerian.

Thus, the vortex will be rather unstable, at least, on the disk's periphery, provided $R_e \to \infty$.

4.2 Magnetic accretion disk

It is assumed that all dependent variables are analytic functions of z in a vicinity of the midplane. Then, in view of the symmetry, we may write:

$$B_\varphi = B_{\varphi 1} z + \ldots,\ B_z = B_{z0} + B_{z2} z^2 + \ldots,\ B_r = B_{r1} z + B_{\varphi 3} z^3 + \ldots,$$

$$\rho = \rho_0 + \rho_2 z^2 + \ldots,\ T = T_0 + T_2 z^2 + \ldots,\ v_\varphi = v_{\varphi 0} + v_{\varphi 2} z^2 + \ldots,$$

$$v_r = v_{r0} + v_{r2} z^2 + \ldots,\ v_z = v_{z1} z + v_{z3} z^3 + \ldots,$$

$$\Phi = -\frac{1}{\sqrt{r^2 + z^2}} = \Phi_0 + \Phi_2 z^2 + \ldots = -\frac{1}{r} + \frac{1}{2r^3} z^2 + \ldots\ . \tag{140}$$

It is also assumed in this section that $B_{z0} \neq 0$.

A steady-state magnetic accretion disk with $v_{r0} \neq 0$, $\rho_0 \neq 0$, and with a negligible dynamic viscosity, i.e. $\mu = 0$, will be our initial concern. The magnetic field will be called as almost poloidal if

$$B_{\varphi 1} = 0 \Rightarrow B_\varphi = O\left(z^3\right), \tag{141}$$

and the magnetic field will be called as almost axial if

$$B_{\varphi 1} = 0,\ B_{r1} = 0 \Rightarrow B_\varphi = O\left(z^3\right),\ B_r = O\left(z^3\right). \tag{142}$$

Let us prove that the circular velocity at the midplane will be the vortex, i.e.

$$v_{\varphi 0} \equiv v_\varphi\big|_{z=0} = \frac{C_{\varphi 0}}{r}, \quad C_{\varphi 0} = const, \tag{143}$$

if and only if the magnetic field will be almost poloidal. Actually, since $\mu = 0$, the steady-state version of Eqs. (15), (17) is the following.

$$\frac{1}{r}\frac{\partial\left(r\rho v_r\right)}{\partial r} + \frac{\partial\left(\rho v_z\right)}{\partial z} = 0, \tag{144}$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\left(\rho v_\varphi v_r - \frac{E_u}{\beta}B_\varphi B_r\right) +$$
$$\frac{\partial}{\partial z}\left(\rho v_\varphi v_z - \frac{E_u}{\beta}B_\varphi B_z\right) + \frac{\rho v_\varphi v_r}{r} - \frac{E_u}{\beta}\frac{B_\varphi B_r}{r} = 0. \tag{145}$$

In view of (144), (145), and (140), we obtain

$$\frac{1}{r}\frac{\partial\left(r\rho_0 v_{r0}\right)}{\partial r} + \rho_0 v_{z1} = 0, \tag{146}$$

$$\frac{v_{\varphi 0}}{r}\frac{\partial}{\partial r}r\left(\rho_0 v_{r0}\right) + \frac{r\left(\rho_0 v_{r0}\right)}{r}\frac{\partial v_{\varphi 0}}{\partial r} - \frac{E_u}{\beta}B_{\varphi 1}B_{z0} + \rho_0 v_{\varphi 0}v_{z1} + \frac{\rho_0 v_{\varphi 0}v_{r0}}{r} = 0. \tag{147}$$

By virtue of (146), we obtain from (147):

$$\rho_0 v_{r0}\frac{\partial v_{\varphi 0}}{\partial r} - \frac{E_u}{\beta}B_{\varphi 1}B_{z0} + \frac{\rho_0 v_{\varphi 0}v_{r0}}{r} = 0. \tag{148}$$

Let the magnetic field will be almost poloidal, i.e. $B_{\varphi 1} = 0$. Then, in view of (148), we have

$$\rho_0 v_{r0}\frac{\partial r v_{\varphi 0}}{r\partial r} = 0. \tag{149}$$

Equality (149) proves (143).

Let (143) be valid. In view of (148), we have

$$\rho_0 v_{r0}\frac{\partial r v_{\varphi 0}}{r\partial r} - \frac{E_u}{\beta}B_{\varphi 1}B_{z0} = 0. \tag{150}$$

By virtue of (143), we find from (150):

$$B_{\varphi 1}B_{z0} = 0. \tag{151}$$

Equality (151) proves (141).

Let us now consider a steady-state magnetic disk with $v_r = 0$ (and, hence, $v_z = 0$) and with a negligible dynamic viscosity, i.e. $\mu = 0$. Let us remind that such assumptions for the case of non-magnetic disk lead to the ill-posed problem in the sense of Hadamard (see Sec. 4.1). In particular, the solution to

the mathematical model (Sec. 4.1) is not unique. We intend to find a steady-state solution to the MHD system (15)-(22) provided $R_e \to \infty$, $\vartheta_{ke} = 0$. In such a case, the MHD system is reduced to the following.

$$-\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{E_u}{\beta}B_r^2\right) - \frac{\partial}{\partial z}\left(\frac{E_u}{\beta}B_r B_z\right) +$$

$$\frac{E_u}{\beta}\frac{B_\varphi^2}{r} - \frac{\rho v_\varphi^2}{r} = -\frac{\partial}{\partial r}\left(E_u P + \frac{E_u}{\beta}\frac{B^2}{2}\right) - \frac{\rho}{F_r}\frac{\partial \Phi}{\partial r}, \tag{152}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{E_u}{\beta}B_\varphi B_r\right) - \frac{\partial}{\partial z}\left(\frac{E_u}{\beta}B_\varphi B_z\right) - \frac{E_u}{\beta}\frac{B_\varphi B_r}{r} = 0, \tag{153}$$

$$\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{E_u}{\beta}B_z B_r\right) + \frac{\partial}{\partial z}\left(\frac{E_u}{\beta}B_z^2\right) =$$

$$\frac{\partial}{\partial z}\left(E_u P + \frac{E_u}{\beta}\frac{B^2}{2}\right) + \frac{\rho}{F_r}\frac{\partial \Phi}{\partial z} \tag{154}$$

$$\frac{\partial\left(v_\varphi B_r\right)}{\partial r} + \frac{\partial\left(v_\varphi B_z\right)}{\partial z} = 0, \tag{155}$$

$$\frac{1}{r}\frac{\partial r B_r}{\partial r} + \frac{\partial B_z}{\partial z} = 0. \tag{156}$$

Eq. (24) is reduced to the following

$$\frac{\partial r\left(v_\varphi B_z\right)}{r \partial r} - \frac{\partial\left(v_\varphi B_r\right)}{\partial z} = 0. \tag{157}$$

Let B_φ be an odd function of z and let $B_r = O\left(z^5\right)$. Then we find from (156) that

$$B_{z0} = B_{z0}\left(r\right),\ B_{z2} = B_{z4} = 0. \tag{158}$$

By virtue of (158), we find from (155) and (153) that

$$v_{\varphi 0} = v_{\varphi 0}\left(r\right),\ v_{\varphi 2} = v_{\varphi 4} = 0,\ B_{\varphi 1} = B_{\varphi 3} = 0. \tag{159}$$

Taking into account that $B_r = O\left(z^5\right)$ and using (158), we obtain from Eq. (154) that (85) is valid for the case of the magnetic disk as well. Hence, using (89), (90), and (158)-(159), we obtain from Eq. (152) that

$$\frac{\rho v_{\varphi 0}^2}{r} = E_u T\frac{\partial \rho}{\partial r} + E_u \rho\frac{\partial T}{\partial r} + \frac{E_u}{\beta}\frac{\partial B_{z0}^2}{\partial r} + \frac{\rho}{F_r}\frac{\partial \Phi}{\partial r} + O\left(z^4\right). \tag{160}$$

Eq. (160) must be valid under all values of z. Assuming $z = H$, we obtain, in view of (84), the following equation in B_{z0}:

$$\frac{E_u}{\beta}\frac{\partial B_{z0}^2}{\partial r} + O\left(\left(z/r\right)^4\right) = 0. \tag{161}$$

Then we find:

$$B_{z0} \approx const. \tag{162}$$

Since $B_r = O\left(z^5\right)$, we obtain from (157), by virtue of (158)-(159), that

$$\frac{\partial r\left(v_{\varphi 0} B_{z0}\right)}{r \partial r} = 0. \tag{163}$$

We find, in view of (162), (163), that

$$v_{\varphi 0} \approx \frac{const}{r}. \tag{164}$$

Analogously, if $B_r = 0$ (and $v_r = 0$), then we find from (152)-(156) that $B_\varphi = 0$ and $B_z = const$. Thus, in view of (157), the vortex

$$v_\varphi = \frac{const}{r} \tag{165}$$

will be the only solution for the circular velocity, and, hence, the motion will be Rayleigh unstable. Let us note, the density and temperature distributions can be easily found from Eqs. (152), (154), which can be written, in view of the foregoing, as the following:

$$\frac{\rho_0 v_\varphi^2}{r} = \frac{\partial}{\partial r}\left(E_u \rho_0 T_0\right) + \frac{\rho_0}{r^2 F_r}, \tag{166}$$

$$\frac{\partial}{\partial z}\left(E_u \rho T\right) + \frac{\rho}{F_r} \frac{z}{\left(r^2 + z^2\right)^{3/2}} = 0, \tag{167}$$

where the circular velocity, v_φ, is calculated from Eq. (165). Notice, Eqs. (166), (167) coincide with Eqs. (81), (82), and, hence, we obtain (89), (90), and (94).

Notice, we have assumed $v_{r0} \neq 0$ in the above-proven assertion that the midplane circular velocity will be the vortex if and only if the magnetic field will be almost poloidal. The following counter-example demonstrates that the condition $v_{r0} \neq 0$ is essential. We consider the case when $v_r = 0$ and $B_z = B_{z0} + O\left(z^4\right)$ with $B_{z0} = const$. In view of (156), we have

$$B_{r1} = \frac{C_{r1}}{r}, \quad C_{r1} = const. \tag{168}$$

Then, by virtue of (168), we obtain from (157) the power-law model:

$$v_{\varphi 0} = \frac{const}{r^{\varkappa}}, \quad \varkappa \equiv 1 - \frac{C_{r1}}{B_{z0}} = const, \tag{169}$$

which is the only solution for the midplane circular velocity in the frame of our assumptions. However, in view of (153), $B_{\varphi 1} = 0$, i.e. the magnetic field is almost poloidal.

4.2.1 Viscous disk

Let us, first, demonstrate that the vortex, (143), will also be the midplane circular velocity if the dynamic viscosity $\mu = const \neq 0$, provided that the magnetic field will be almost axial. Actually, if $B_\varphi = O\left(z^3\right)$ and $B_r = O\left(z^3\right)$, then, in view of (140), we obtain from the steady-state version of Eq. (17) that

$$\rho_0 v_{r0} \frac{\partial r v_{\varphi 0}}{r \partial r} = \frac{1}{R_e} \frac{\partial}{\partial r} \left[\mu r \frac{\partial}{\partial r} \left(\frac{v_{\varphi 0}}{r} \right) \right] + \frac{2 \mu v_{\varphi 2}}{R_e} + \frac{2\mu}{R_e} \frac{\partial}{\partial r} \left(\frac{v_{\varphi 0}}{r} \right). \tag{170}$$

In view of (22) and (140), we have

$$\frac{1}{r} \frac{\partial r B_{r1}}{\partial r} + 2 B_{z2} = 0. \tag{171}$$

Hence

$$B_{z2} = 0. \tag{172}$$

By virtue of (140), we find from (20) that

$$\frac{\partial \left(v_{r0} B_{\varphi 1} - v_{\varphi 0} B_{r1} \right)}{\partial r} + 2 \left(v_{z1} B_{\varphi 1} - v_{\varphi 2} B_{z0} - v_{\varphi 0} B_{z2} \right) = 0. \tag{173}$$

Hence, in view of (172), we obtain:

$$v_{\varphi 2} B_{z0} = 0 \Rightarrow v_{\varphi 2} = 0. \tag{174}$$

By virtue of (170) and (174), we find that

$$\rho_0 v_{r0} \frac{\partial r v_{\varphi 0}}{r \partial r} = \frac{\mu}{R_e} \frac{\partial}{\partial r} \left[r \frac{\partial}{\partial r} \left(\frac{v_{\varphi 0}}{r} \right) + 2 \frac{v_{\varphi 0}}{r} \right]. \tag{175}$$

Obviously, the vortex, (143), fulfills Eq. (175). To estimate other solutions, $v_{\varphi 0}$, to Eq. (175), we will use (116), i.e.

$$v_{z1} = \frac{v_{r0}}{r} + O\left(\left(H \diagup r \right)^2 \right). \tag{176}$$

By virtue of (176), we obtain from (103) that

$$\frac{\partial \left(r \rho_0 v_{r0} \right)}{r \partial r} + \frac{\rho_0 v_{r0}}{r} + O\left(\left(H \diagup r \right)^2 \right) = 0. \tag{177}$$

Hence, we find with accuracy $O\left(\left(H \diagup r \right)^2 \right)$ that

$$r \rho_0 v_{r0} = -\frac{Q_0}{r}, \quad Q_0 = const \geq 0. \tag{178}$$

Since $\mu = const \neq 0$, we assume that $\mu = 1$, without loss in generality. Then, by virtue of (178), we rewrite (175) to read

$$-\frac{a_r}{r^3} \frac{\partial r^2 \Omega_0}{\partial r} = \frac{\partial}{\partial r} \left(r \frac{\partial \Omega_0}{\partial r} + 2 \Omega_0 \right), \quad a_r \equiv Q_0 R_e = const, \tag{179}$$

where Ω_0 denotes the angular velocity at $z = 0$. We will investigate (179) in the vicinity of $r = 1$. Let $x = \ln r$, then we have instead of (179)

$$\frac{\partial^2 \Omega_0}{\partial x^2} + [2 + a_r \exp(-x)] \frac{\partial \Omega_0}{\partial x} + 2a_r \exp(-x) \Omega_0 = 0. \tag{180}$$

In the vicinity of $r = 1$, i.e. in the vicinity of $x = 0$, we obtain, in view of (180), that

$$\Omega_0 \approx \frac{C_1}{r^2} + \frac{C_2}{r^{a_r}}, \quad C_1, C_2 = const. \tag{181}$$

Thus, if $a_r \geq 2$, i.e. if R_e will be sufficiently big, then the motion will be Rayleigh unstable, and, hence, the motion can be stable under sufficiently small Reynolds number. In all likelihood, it will be a turbulent motion.

As indicated above, the midplane circular velocity can be the vortex if the dynamic viscosity $\mu = const$. Let us now consider the case when the midplane circular velocity is not the vortex, but the power-law model:

$$v_{\varphi 0} = \frac{C_\varphi}{r^\varkappa}, \quad \varkappa = const \neq 1, \; C_\varphi = const. \tag{182}$$

The power-law model, (182), can fulfill Eq. (175) if $\mu = \mu(r)$. The following inequality must be valid to fulfil (96).

$$\varkappa > 0.5. \tag{183}$$

It is also assumed that the magnetic field will be almost axial, i.e. $B_\varphi = O\left(z^3\right)$ and $B_r = O\left(z^3\right)$. To estimate $\mu = \mu(r)$ at $z = 0$ we assume $v_{r0} = 0$. We obtain from the steady-state version of Eq. (17) that

$$\frac{1}{R_e}\frac{\partial}{\partial r}\left[\mu r \frac{\partial}{\partial r}\left(\frac{v_{\varphi 0}}{r}\right)\right] + \frac{2\mu}{R_e}\frac{\partial}{\partial r}\left(\frac{v_{\varphi 0}}{r}\right) = 0. \tag{184}$$

Then, by virtue of (182), we find from (184) that

$$\mu = \frac{C_\mu}{r^{1-\varkappa}}, \quad C_\mu = const. \tag{185}$$

Assuming the characteristic quantity, μ_*, for viscosity such that $C_\mu = 1$ and using (113), we represent the viscosity, μ, in the following form:

$$\mu = T^{1-\varkappa}. \tag{186}$$

The power law (e.g., [4], [16], [23]) for the laminar viscosity, μ_l, of dilute gases can be written as the following

$$\mu_l = T^\theta, \tag{187}$$

where typically $\theta = 0.76$ [4]. It is, also, assumed that $\theta = 8/9$ if $90 < T < 300$ $^\circ K$, and $\theta = 0.75$ if $250 < T < 600$ $^\circ K$. If we assume that the flow in question

is laminar, then $\mu = \mu_l$, and, hence, $\varkappa = 1 - \theta$. Since we consider the flow at the periphery of the disk, i.e. under a low temperature, we find that

$$\varkappa < 0.25. \tag{188}$$

The inequality (188) contradicts to (183), and, hence, the flow in question is rather turbulent.

The Euler number characterizes "losses" in a flow [10], and it is higher in a turbulent flow than that in the laminar regime [3]. Let us now estimate the value of $\varkappa$ in (182) by assuming that the turbulent gas tends to flow with minimal losses, i.e. to have the Euler number as small as possible. Using Prandtl and Kolmogorov suggestion [1, p. 230] that the turbulent viscosity, μ, is proportional to the square root of the kinetic energy of turbulence, $\overline{\kappa}$, we evaluate $\mu_0 \equiv \mu|_{z=0}$ as

$$\mu_0 = C_\kappa L_\kappa \rho_0 \overline{\kappa}^{0.5}, \quad C_\kappa, L_\kappa = const. \tag{189}$$

By virtue of (113), (101), and (182), we rewrite (94) to read:

$$\rho_0 = \frac{\rho_0^0}{r^{\alpha-1}} \exp \zeta \left[\frac{1}{r^{2\varkappa-1}} - 1 \right], \quad \zeta = \frac{C_\varphi^2}{E_u \left(2\varkappa - 1\right)}, \ \alpha = \frac{1}{E_u F_r}. \tag{190}$$

Let the kinetic energy of turbulence $\overline{\kappa} = const$. In such a case, we obtain from (189) that

$$\mu_0 = \frac{C_\rho}{r^{\alpha-1}} \exp \zeta \left[\frac{1}{r^{2\varkappa-1}} - 1 \right], \quad C_\rho \equiv \rho_0^0 C_\kappa L_\kappa \overline{\kappa}^{0.5} = const. \tag{191}$$

Equating (185) and (191) at $r = 1$, we find that $C_\rho = C_\mu$. Let us now assume that μ of (185) and μ_0 of (191) coincide each other in the vicinity of $r = 1$, i.e. at $r = 1 + \varepsilon$ ($\varepsilon \ll 1$), with accuracy $O\left(\varepsilon^2\right)$. In such a case we obtain that

$$E_u \propto \frac{1}{2 - \varkappa}. \tag{192}$$

As we can see from (192) and (183), $E_u \to \min$ if $\varkappa \to 0.5$. Thus, in the frame of our assumptions, we find that the turbulent flow tends to be Keplerian.

5 Concluding remarks

On the basis of the exact solution to the MHD system in Sec. 2 we may conclude that vertically isothermal disks will, in fact, be totally isothermal under the assumption that the radial components, v_r and B_r, of velocity and magnetic field, respectively, are negligible. Furthermore, the disks cannot be considered as thin in terms of, e.g., [6] even if the Euler number $0 < E_u \ll 1$.

The exact solution to the MHD system in Sec. 3 corroborates the view [20] that thin disk accretion must be highly nonadiabatic. Despite of the fact that adiabatic disks (see Sec. 3) are more trustworthy than isothermal ones, we find that the non-dimensional semi-thickness $H \propto r^2$ instead of the ratio $H \propto r$ [20].

The exact solutions to the MHD systems in Sec. 2 and Sec. 3 prove that the vortex will be the only solution for the circular velocity provided that the flow is charge-neutral. Let us note that the exact solutions are found under the assumption that $v_r = 0$ and $B_r = 0$.

The approach developed in Sec. 4 for the modeling of thin accretion disks turns out to be efficient. In the case of non-magnetic disk, this approach enables to obtain, with ease: the solution for the steady-state non-viscous disk with a good accuracy, to find the non-dimensional semi-thickness $H \propto r$, to prove that all solutions for the midplane circular velocity are unstable provided the disk is non-viscous. Using this approach one can prove that the midplane circular velocity will be the vortex if and only if the magnetic field will be almost poloidal. The approach of Sec. 4 enables one to demonstrate that the pure hydrodynamic turbulence in accretion disks is possible, and to demonstrate that the turbulent flow tends to be Keplerian.

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